ON THE REPRESENTATION OF THE GENERAL SOLUTION OF THE BASIC EQUATIONS OF THE STATIC THEORY OF ELASTICITY FOR AN ISOTROPIC BODY WITH THE AID OF HARMONIC FUNCTIONS

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The known representations of scalar biharmonic function in a three-dimensional domain by means of three-dimensional harmonic functions are considered jointly as a general representation of a biharmonic function through harmonic ones, independently of the problem regarding the possibility of reducing the number of such functions. This representation is supplemented by an expression containing plane harmonic functions.

In employing this expression for the construction of the general solution of the basic equations of the theory of static elasticity for an isotropic body, it is possible to obtain a generalized solution of Papkovich, which includes two of his separate propositions and which is supplemented by a solution using plane functions, which do not follow immediately from the other ones. One of the two methods indicated by him, which leads to a differential relationship between harmonic functions, may be represented in a form which contains independent functions.

1. On the representation of biharmonic through harmonic functions. (a) We assume that a biharmonic scalar function B, considered in a three-dimensional domain, may be represented in the form of a product of two scalar functions R and S, which must be four times differentiable in this domain:

$$B = RS \tag{1.1}$$

Then the condition for the function B to be biharmonic may be represented in the form:

$$\begin{aligned} R(\triangle\triangle S) + 4(\nabla R)\cdot(\nabla\triangle S) + 4\,(\nabla^2 R)\,..\,(\triangle^2 S) + \\ + 2\triangle(R)\,(\triangle S) + 4\,(\nabla\triangle R)\cdot(\nabla S) + (\triangle\triangle R)\,S = 0 \end{aligned} \tag{1.2}$$

Here is Hamilton's operator (nabla), Δ is Laplace's operator, and the dot is the symbol for scalar multiplication. In order that the given equation is satisfied for arbitrary functions R and S, it is necessary that each additive term on its left-hand side be equal to zero separately.

Excluding from our considerations the trivial case when one of the functions, R or S, is an arbitrary biharmonic one, and the other is a constant, it is easy to conclude that the indicated requirement of vanishing of each additive term will be satisfied for most general functions R and S, in the following four cases:

$$\begin{array}{llll} 1) & \nabla R = 0, \quad \nabla \Delta S = 2\mathbf{b}; & 3) & \nabla^2 R = 0, \quad \Delta S = 2c \\ 2) & \nabla R = \mathbf{b}, \quad \nabla \Delta S = 0; & 4) & \nabla^2 R = 2c\mathbf{I}, \quad \Delta S = 0 \end{array} \tag{1.3}$$

where $\mathbf{b}$ is a constant vector, c is a constant scalar and $\mathbf{I}$ is the unit tensor of rank two. Obviously, in view of the symmetry of the equation (1.2) with regard to functions R and S, their properties may be interchanged.

Designating by Φ a harmonic function in the three-dimensional domain considered and indicating by K and L the constant tensors of rank two and three, respectively, the results of integration of equations (1.3) may be represented as follows:

$$\begin{array}{lll} 1) & R = a, & S = \Phi + K \ldots \mathbf{rr} + L \ldots \mathbf{rrr} \\ 2) & R = a\,\mathbf{b}\cdot\mathbf{r}, & S = \Phi + K \,..\, \mathbf{rr} \\ 3) & R = a + \mathbf{b}\cdot\mathbf{r}, & S = \Phi + K \,..\, \mathbf{rr} \\ 4) & R = a + \mathbf{b}\cdot\mathbf{r} + cr^2, & S = \Phi \\ & (L^{(1;2)} + L^{(2;3)} + L^{(3;1)} = \mathbf{b}, & K^\circ = c) \end{array} \tag{1.4}$$

where a is a constant vector, $\mathbf{r}$ is the position vector of the point under consideration, r is its modulus, K^0 is the scalar contraction of the tensor K of rank two, $L^{(1,2)}$ is the scalar contraction of the tensor L of rank three with respect to its first and second ranks; analogously, $L^{(2,3)}$ and $L^{(3,1)}$ indicate the scalar contraction of tensor L with respect to its second and third ranks, and with respect to its third and first ranks, respectively.

We shall use the results of the fourth case since it will be shown later that the results of the remaining cases can be reduced to it. The function B in this case, in accordance with (1.1) and as a consequence of representation of the fourth group of equations (1.4), will be of the form:

$$B = (a + \mathbf{r}\cdot\mathbf{b} + r^2 c)\,\Phi \tag{1.5}$$

Obviously, a more general formula for the biharmonic function may be obtained by adding such representations as (1.5), in which the constants and the value of the harmonic function Φ are changed.

Designating such variable constants and functions by the subscript n, we introduce harmonic scalar functions and a harmonic vector function

$$F = \sum_n a_n \Phi_n, \qquad H = \sum_n c_n \Phi_n, \qquad \mathbf{G} = \sum_n \mathbf{b}_n \Phi_n$$

The biharmonic function B may then be written as

$$B = F + \mathbf{r}\cdot\mathbf{G} + r^2 H \tag{1.6}$$

Returning to the first three groups of equations (1.4), it should be noted that in using them for the construction of the biharmonic function in accordance with formula (1.1), there occur also the products $a(\mathbf{r}\,\mathbf{r}\,..\,K + \mathbf{r}\,\mathbf{r}\,\mathbf{r}\,...\,L)$ and $(a + \mathbf{r}\cdot\mathbf{b})(\mathbf{r}\,\mathbf{r}\,..\,k)$, which represent rational polynomials of second and third degree with respect to the variables, as may be easily verified, which may be obtained with the aid of formula (1.6).

(b) Let the three-dimensional harmonic function S be of the form:

$$S = \Phi_1 + z\Phi_2 \tag{1.7}$$

where Φ_1 and Φ_2 are two harmonic functions at the point of a domain in a certain plane, and z is measured along the normal to this plane.

If this expression is substituted into (1.2), then it takes the form:

$$4(\nabla^2 R)\,..\,[\nabla^2\Phi_1 + z(\nabla^2\Phi_2) + \mathbf{k}(\nabla\Phi_2) + (\nabla\Phi_2)\mathbf{k}] +$$
$$+ 4(\nabla\Delta R)\,.\,[\nabla\Phi_1 + z(\nabla\Phi_2) + \Phi_2\mathbf{k}] + (\Delta\Delta R)(\Phi_1 + z\Phi_2) = 0$$

where $\mathbf{k}$ is in the z-direction; in the absence of such a point, or if the vectors are oblique, the multiplication has to be considered as a dyadic one.

The given equation will be satisfied independently of the particular form of the plane harmonic functions Φ and Φ, if

$$\nabla^2 R = 2[c(\mathbf{I} - \mathbf{kk}) + g\mathbf{kk}]$$

where c and g are two constant scalars. From this follows the expression:

$$R = a + \mathbf{r}\cdot\mathbf{b} + \rho^2 c + z^2 g$$

where a, $\mathbf{b}$ and $\mathbf{r}$ take on the same values as earlier in formula (1.5), and ρ is the modulus of the position vector of the point in the plane normal to the z-axis, i.e., in a plane in which the arguments of functions Φ_1 and Φ_2 vary.

The harmonic function B to be found will be in this case, as a consequence of (1.1)

$$B = (a + \mathbf{r}\cdot\mathbf{b} + \rho^2 c + z^2 g)(\Phi_1 + z\Phi_2)$$

Adding such expressions with different values for the constants and for the functions Φ_1 and Φ_2, B may be expressed by the equation:

$$B = F_0 + \mathbf{r}\cdot\mathbf{G}_1 + z\mathbf{r}\cdot\mathbf{G}_2 + \rho^2 P_1 + z\rho^2 P_2 + z^3 P_3 \tag{1.8}$$

Here F_0, P_1, P_2 and P_3 are plane, mutually independent harmonic scalar functions and $\mathbf{G}_1$ and $\mathbf{G}_2$ are harmonic vector functions, whose three components along the coordinates are represented by plane scalar functions. If the above expression is compared with the representation (1.6), it may be noted that some of the additive terms of formula (1.8) do not follow immediately from equation (1.6).

In fact, putting, for example, $H = z\Phi_2$, we find

$$r^2 H = (z\rho^2 + z^3)\Phi_2$$

where the decomposition of the right-hand side into two mutually independent additive terms, like the two last ones in formula (1.8), does not take place. Neither are they decomposed with the aid of other additive terms of the representation (1.6).

(c) Seeking to obtain an expression of the three-dimensional biharmonic function through harmonic ones, which would enter differently into such an expression, we supplement formula (1.6) by additive terms which would ascertain a direct occurrence of therms which enter into equation (1.8).

Keeping in mind that expressions of the type (1.8) may be constructed separately for each of the three mutually perpendicular planes of three-dimensional space, the supplementary additive term, which shall be designated by B_p, will be given the form

$$B_p = \mathbf{rrr}\ldots P \equiv \mathbf{r}^3\ldots P$$

where P is a three-component tensor of rank three of special construction represented by formula

$$P = P_x\mathbf{iii} + P_y\mathbf{jjj} + P_z\mathbf{kkk} \tag{1.9}$$

Here P_x, P_y and P_z are three harmonic functions in certain domains which are situated on mutually orthogonal planes, whose normals are indicated by subscripts, while $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors of the Cartesian system of coordinates.

After having supplemented formulas (1.6) by this expression, we obtain the following representation of the biharmonic function:

$$B = F + \mathbf{r}\cdot\mathbf{G} + r^2 H + \mathbf{r}^3\ldots P \tag{1.10}$$

In Cartesian coordinates this expression will take on the form:

$$B = F + xG_x + yG_y + zG_z + (x^2 + y^2 + z^2) H + \\ + x^3 P_x(y, z) + y^3 P_y(z, x) + z^3 P_z(x, y)$$

where G_x, G_y and G_z are the coordinate components of vector $\mathbf{G}$.

Since, in the solution of concrete problems it may be desirable to have at our disposal different forms of additive terms in the right-hand side of equation (1.10), we shall conserve in this equation all the additive terms, as we proceed to the general constructions in three-dimensional domains.

2. General solution of the basic equations of the theory of elasticity. As is known, the problem concerning the determination of displacements and stresses which are produced in an isotropic elastic body, subjected to the action of external body and surface forces, may be reduced to the problem concerning the state of stress in the same body in the absence of body forces, but subjected to some different surface tractions. This possibility of elimination of body forces in the solution of the basic differential equations of the theory of elasticity will be used here.

We start out from the following representation of the displacement vector in an elastic isotropic body:

$$\mathbf{u} = \mathbf{v} - \nabla B \tag{2.1}$$

where $\mathbf{v}$ is a harmonic vector and B is a biharmonic scalar. From the differential equation of equilibrium

$$(1 - 2\nu)\triangle \mathbf{u} + \nabla^2 \cdot \mathbf{u} = 0$$

where ν is Poisson's ratio, it follows that the functions $\mathbf{v}$ and B should be related by the expression

$$\nabla \cdot \mathbf{v} = 2(1 - \nu) \triangle B \tag{2.2}$$

The expressions for the displacement components along the coordinates, analogous to the representation (2.1), for the case in which the vector $\mathbf{v}$ had only one component, were used already by Hertz [1]. For the plane problem of the theory of elasticity such formulas were obtained in their complete form by Love [2].

Let us use equation (1.10) to represent the biharmonic function B. In this case equation (2.2) will take on the form:

$$\nabla \cdot \mathbf{v} = 4(1 - \nu)[\nabla \cdot \mathbf{G} + 3H + 2\mathbf{r} \cdot (\nabla H) + 3\mathbf{r} \cdot P^{(1;2)}] \tag{2.3}$$

Let us now represent the vector $\mathbf{v}$ as:

$$\mathbf{v} = 4(1 - \nu)(\mathbf{G} + \mathbf{w}) \tag{2.4}$$

where $\mathbf{w}$ is also a harmonic vector; we find then from the preceding equation, that the vector $\mathbf{w}$ should satisfy the equation

$$\Delta \cdot \mathbf{w} = 3H + \mathbf{r} \cdot (\nabla H) + 3\mathbf{r} \cdot P^{(1;2)} \tag{2.5}$$

The displacement vector $\mathbf{u}$, represented by equation (2.1), may also, in consequence of expressions (2.4) and (2.10), be represented as:

$$\mathbf{u} = 4(1-\nu)(\mathbf{G} + \mathbf{w}) - \nabla(F + \mathbf{r}\cdot\mathbf{G} + r^2H + \mathbf{r}^3 \ldots P) \tag{2.6}$$

The given equation represents a generalized solution of Papkovich, inasmuch as it includes two forms of the solution which were suggested by him separately and which are supplemented by additive terms consisting of plane harmonic functions which he did not take into account.

In fact, if all the functions on the right-hand side of this equation vanish, except F and $\mathbf{G}$, we obtain the representation

$$\mathbf{u} = 4(1-\nu)\mathbf{G} - \nabla(F + \mathbf{r}\cdot\mathbf{G}) \tag{2.7}$$

which is known in the literature as the Papkovich-Neuber solution, established by Grodskii [3,4,5,6] as well as these authors. A particular form of this solution for the case when two coordinate components are absent in the harmonic vector $\mathbf{G}$, was obtained earlier by Boussinesq [7].

If, on the right-hand side of equation (2.6), all the functions are eliminated, except $\mathbf{w}$, F and H, then we obtain the formula

$$\mathbf{u} = 4(1-\nu)\mathbf{w} - \nabla(F + r^2H) \tag{2.8}$$

which was also suggested, independently from the previous one, by Papkovich as a solution of the basic equations of the theory of elasticity [4].

The relationship between the harmonic vector $\mathbf{w}$ and the function H, by virtue of (2.5), will be

$$\nabla\cdot\mathbf{w} = 3H + 2r\frac{\partial H}{\partial r} \tag{2.9}$$

which was also indicated by him. However, the integral of this equation was not obtained by him.

The question regarding the possibility of eliminating one or the other function in the first form of the solution (2.7) was a subject of special studies.

Neuber has shown how one of the components of vector $\mathbf{G}$ or the scalar F may be eliminated from the general expression for the displacement vector. Papkovich [4] established that the scalar function F may be omitted only in the case when $\nu \neq 0.25$. Finally, Slobodianskii found that the question regarding the possibility of elimination of the function F depends also on the degree of boundedness of the volume occupied by the body considered and on the presence of internal closed boundary surfaces [8].

The functions which enter into (2.6) are not completely independent, since some of them have to satisfy equation (2.5). It is possible, however, to free oneself from this dependence by means of integration. With this aim in mind, we introduce a harmonic vector **R**, determined by

$$H = \nabla\cdot\mathbf{R} \tag{2.10}$$

and a harmonic tensor Q of rank four in the form

$$Q = \mathbf{iii}\,[Q_{xy}(y,z)\,\mathbf{j} + Q_{xz}(y,z)\,\mathbf{k}] + \mathbf{jjj}\,[Q_{yz}(z,x)\,\mathbf{k} + Q_{yx}(z,x)\,\mathbf{i}] + \\ + \mathbf{kkk}\,[Q_{zx}(x,y)\,\mathbf{i} + Q_{zy}(x,y)\,\mathbf{j}] \tag{2.11}$$

where $Q_{xy}(y, z)$, $Q_{xz}(y, z)$ etc., are functions of the two Cartesian coordinates of the point indicated; this tensor is related to the tensor P by the condition

$$P = Q\cdot\nabla \tag{2.12}$$

It is easily verified that the integral of equation (2.5) may be represented in this case as

$$\mathbf{w} = \mathbf{r}(\nabla\cdot R) + \mathbf{r}\cdot(\nabla\mathbf{R}) - (\nabla\mathbf{R})\cdot\mathbf{r} + 3\mathbf{r}\cdot Q^{(1;2)} + \frac{1}{4(1-\nu)}(\nabla S + \nabla\times\mathbf{T}) \tag{2.13}$$

where S and $\mathbf{T}$ are arbitrary harmonic functions in the domain considered, a scalar and a vector function, respectively, and $\times$ is the symbol for vector multiplication.

Formula (2.6) for the displacement vector then takes the form:

$$\mathbf{u} = \nabla F + \nabla\times\mathbf{T} + 4(1-\nu)[\mathbf{G} + \mathbf{r}\cdot(\nabla\cdot\mathbf{R}) + \mathbf{r}\cdot(\nabla\mathbf{R}) - (\nabla\mathbf{R})\cdot\mathbf{r} + 3\mathbf{r}\cdot Q^{(1;2)}] - \\ - \nabla[\mathbf{r}\cdot\mathbf{G} + r^2(\nabla\cdot\mathbf{R}) + \mathbf{r}^3\ldots(Q\cdot\nabla)] \tag{2.14}$$

where $S -\ F$ is combined into $+\ F$, or also in the form:

$$\mathbf{u} = \nabla F + \nabla\times\mathbf{T} + (3-4\nu)\mathbf{G} - (\nabla\mathbf{G})\cdot\mathbf{r} + 2(1-2\nu)\mathbf{r}(\nabla\cdot\mathbf{R}) + \\ + 4(1-\nu)[\mathbf{r}\cdot(\nabla\mathbf{R}) - (\nabla\mathbf{R})\cdot\mathbf{r}] - r^2(\nabla^2\cdot\mathbf{R}) + \\ + 12(1-\nu)\mathbf{r}\cdot Q^{(1;2)} - 3\mathbf{r}^2\ldots(Q\cdot\nabla) - \mathbf{r}^3\ldots(Q\cdot\nabla^2) \tag{2.15}$$

In a Cartesian coordinate system the component u_x of the vector **u** may be represented in this case as

$$u_x = \frac{\partial F}{\partial x} + \frac{\partial T_z}{\partial y} - \frac{\partial T_y}{\partial z} + (3-4\nu)G_x - \left(x\frac{\partial G_x}{\partial x} + y\frac{\partial G_y}{\partial x} + z\frac{\partial G_z}{\partial x}\right) + \\ + 2(1-2\nu)x\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z}\right) + 4(1-\nu)\left[z\left(\frac{\partial R_x}{\partial z} - \frac{\partial R_z}{\partial x}\right) + \right. \\ \left. + y\left(\frac{\partial R_x}{\partial y} - \frac{\partial R_y}{\partial x}\right)\right] - (x^2+y^2+z^2)\frac{\partial}{\partial x}\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z}\right) + \\ + 12(1-\nu)[yQ_{yx}(z,x) + zQ_{zx}(x,y)] - 3x^2\left(\frac{\partial Q_{xy}}{\partial y} + \frac{\partial Q_{xz}}{\partial z}\right) - \\ - y^3\frac{\partial}{\partial x}\left(\frac{\partial Q_{yz}}{\partial z} + \frac{\partial Q_{yx}}{\partial x}\right) - z^3\frac{\partial}{\partial x}\left(\frac{\partial Q_{zx}}{\partial x} + \frac{\partial Q_{zy}}{\partial y}\right) \tag{2.16}$$

The corresponding expressions for the components u_y and u_z for the same vector are obtained by cyclic permutation of coordinates and subscripts.

It is clear that the representation (2.14) or (2.15) includes also the second solution (2.8) of Papkovich in a form which is free of any interrelation among the harmonic functions entering into it.

If the stress tensor σ is introduced by means of Hooke's law

$$\sigma = \frac{E}{2(1-\nu)}\left(\nabla\mathbf{u}+\mathbf{u}\nabla+\frac{2\nu}{1-2\nu}\Delta\cdot\mathbf{u}\mathbf{I}\right)$$

where E is the elastic modulus, then the following representation for this tensor is obtained, using equation (2.15)

$$\begin{gathered}\sigma = \frac{E}{2(1-\nu)}\{2\nabla^2 F + \nabla^2\times\mathbf{T} - \mathbf{T}\times\nabla^2 + 2(1-2\nu)(\nabla\mathbf{G}+\mathbf{G}\nabla) - \\ - 2(\nabla^2 G)\cdot\mathbf{r} + 4\nu(\nabla\cdot\mathbf{G})\mathbf{I} + 4(1+\nu)(\nabla\cdot\mathbf{R})\mathbf{I} + 8\nu\mathbf{r}\cdot(\nabla^2\cdot\mathbf{R})\mathbf{I} + \\ + 4(1-\nu)[\mathbf{r}\cdot(\nabla^2\mathbf{R}) + (\mathbf{R}\nabla^2)\cdot\mathbf{r} - 2(\nabla^2\mathbf{R})\cdot\mathbf{r}] - 2r^2(\nabla^3\cdot\mathbf{R}) - \\ - 4\nu[(\nabla^2\cdot\mathbf{R})\mathbf{r} + \mathbf{r}(\nabla^2\cdot\mathbf{R})] + 12\nu\mathbf{r}\cdot(Q^{(1;2)}\cdot\triangle)\mathbf{I} - 12\mathbf{r}\cdot(Q\cdot\nabla) - \\ - 2\mathbf{r}^3\cdot\cdot(Q\cdot\nabla^3) + 12(1-\nu)[Q^{(1;2)} + \underline{Q}^{(1;2)} + \mathbf{r}(Q^{(1;2)}\nabla) + \underline{\mathbf{r}\cdot(Q^{(1;2)}\nabla)}] - \\ - 6[\mathbf{r}^2\cdot\cdot(Q\cdot\triangle^2) + \underline{\mathbf{r}^2\cdot\cdot(Q\cdot\nabla^2)}]\}\end{gathered} \qquad (2.17)$$

Here, in the underlined tensorial expressions, transposition of dyadic factors is assumed, or transposition of subscripts in coordinate representations.

We give below the expressions for the coordinate components of the tensor σ in a Cartesian system which follow from the equation (2.17):

$$\begin{gathered}\sigma_{xx} = \frac{E}{1+\nu}\left\{\frac{\partial^2 F}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial T_z}{\partial y} - \frac{\partial T_y}{\partial z}\right) + 2(1-\nu)\frac{\partial G_x}{\partial x} + \right.\\ + 2\nu\left(\frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right) - \left(x\frac{\partial^2 G_x}{\partial x^2} + y\frac{\partial^2 G_y}{\partial x^2} + z\frac{\partial^2 G_z}{\partial x^2}\right) + \\ + 2(1+\nu)\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z}\right) + 4\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left[\frac{\partial R_x}{\partial x} + \nu\left(\frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z}\right)\right] - \\ - 4(1-\nu)\left(y\frac{\partial^2 R_y}{\partial x^2} + z\frac{\partial^2 R_z}{\partial x^2}\right) - (x^2+y^2+z^2)\frac{\partial^2}{\partial x^2}\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z}\right) - \\ - 6(1-\nu)x\left(\frac{\partial Q_{xy}}{\partial y} + \frac{\partial Q_{xz}}{\partial z}\right) + 6\nu\left(y\frac{\partial Q_{yz}}{\partial z} + z\frac{\partial Q_{zy}}{\partial_y}\right) + \qquad (2.18) \\ \left. + 6(2-\nu)\left(y\frac{\partial Q_{yx}}{\partial x} + z\frac{\partial Q_{zx}}{\partial x}\right) - \left[y^3\frac{\partial^2}{\partial x^2}\left(\frac{\partial Q_{yz}}{\partial z} + \frac{\partial Q_{yx}}{\partial x}\right) + z^3\frac{\partial^2}{\partial x^2}\left(\frac{\partial Q_{zx}}{\partial x} + \frac{\partial Q_{zy}}{\partial y}\right)\right]\right\}\end{gathered}$$

$$\sigma_{yz} = \frac{E}{1+\nu}\left\{\frac{\partial^2 F}{\partial y\partial z} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial T_y}{\partial x} - \frac{\partial T_x}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial T_x}{\partial z} - \frac{\partial T_z}{\partial x}\right) + \right.$$

$$+(1-2\nu)\left(\frac{\partial G_z}{\partial y}+\frac{\partial G_y}{\partial z}\right)-\left(x\frac{\partial^2 G_x}{\partial y\partial z}+y\frac{\partial^2 G_y}{\partial y\partial z}+z\frac{\partial^2 G_z}{\partial y\partial z}\right)-$$
$$-4(1-\nu)x\frac{\partial^2 R_x}{\partial y\partial z}-2\nu\left(y\frac{\partial}{\partial z}+z\frac{\partial}{\partial y}\right)\frac{\partial R_x}{\partial x}-2\left(z\frac{\partial^2 R_z}{\partial y\partial z}+y\frac{\partial^2 R_y}{\partial y\partial z}\right)-$$
$$-2\nu\left(y\frac{\partial^2 R_z}{\partial z^2}+z\frac{\partial^2 R_y}{\partial y^2}\right)+2(1-\nu)\left[x\frac{\partial}{\partial x}\left(\frac{\partial R_z}{\partial y}+\frac{\partial R_y}{\partial z}\right)+\right.$$
$$\left.+y\frac{\partial^2 R_z}{\partial y^2}+z\frac{\partial^2 R_y}{\partial z^2}\right]-(x^2+y^2+z^2)\frac{\partial^2}{\partial y\partial z}\left(\frac{\partial R_x}{\partial x}+\frac{\partial R_y}{\partial y}+\frac{\partial R_z}{\partial z}\right)+$$
$$+6(1-\nu)\left[Q_{yz}+Q_{zy}+x\left(\frac{\partial Q_{xz}}{\partial y}+\frac{\partial Q_{xy}}{\partial z}\right)\right]-3\left[y^2\frac{\partial}{\partial z}\left(\frac{\partial Q_{yz}}{\partial z}+\frac{\partial Q_{yx}}{\partial x}\right)+\right.$$
$$\left.+z^2\frac{\partial}{\partial y}\left(\frac{\partial Q_{zx}}{\partial x}+\frac{\partial Q_{zy}}{\partial y}\right)\right]-x^2\frac{\partial^2}{\partial y\partial z}\left(\frac{\partial Q_{xy}}{\partial y}+\frac{\partial Q_{xz}}{\partial z}\right)\Bigg\} \qquad (2.19)$$

The remaining expressions are obtained from the above by cyclic permutation of coordinates and subscripts.

The availability of the general expressions for the displacement vector and the stress tensor given above, which contain harmonic functions in various combinations, will, by means of appropriate selection of these functions, permit these expressions to be made more suitable to the special features of concrete problems.

BIBLIOGRAPHY

1. Hertz, H., Über die Berührung fester elastischer Körper. *J. Reine Angew. Math.* Bd. 92, S. 156-171, 1882; *Ges. Werke Leipzig* Bd. 1, S. 155, 1895.

2. Love, A., *Mathematical Theory of Elasticity*. p. 216, ONTI, 1935.

3. Papkovich, P.F., Vyrazhenie obshchego integrala osnovnykh uravnenii teorii uprugosti cherez garmonicheskie funktsii (An expression for the general integral of the basic equation of the theory of elasticity by means of harmonic functions). *Izv. Akad. Nauk USSR Otd. Mat. Est. Nauk* pp. 1425-1435, 1932.

4. Papkovich, P.F., Obzor nekotorykh obshchikh reshenii differentsial'nykh uravnenii pokoia izotropnogo uprugogo tela (A survey of certain general solutions of differential equations of equilibrium of isotropic elastic body). *PMM* Vol. 1, pp. 117-123, 1937.

5. Grodskii, G.D., Integrirovanie obshchikh uravnenii ravnovesiia izotropnogo uprugogo tela (Integration of the general solutions of equilibrium of isotropic elastic body). *Izv. Akad. Nauk USSR Otd. Mat.Est. Nauk* No. 4, pp. 587-614, 1935.

6. Neuber, H., Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie. *Z. angew. Math. Mech.* Bd. 14, H. 4, S. 203-212, 1934.

7. Boussinesq, J., *Application des Potentials à l'Etude de l'Equilibre et du Mouvement des Solides Elastiques.* Paris, 1885.

8. Slobodianskii, M.G., Obshchie formy reshenii uravnenii uprugosti dlia odnosviaznykh i mnogosviaznykh oblastei, vyrazhennye cherez garmonicheskie funktsii (The general forms of solutions of the equations of elasticity for simply connected and multiply connected domains expressed through harmonic functions). *PMM* Vol. 18, No. 1, pp. 55-74, 1954.

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